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The Laws of Thought and the Laws of Truth as Two Sides of One Coin

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Abstract

Some think that logic concerns the "laws of truth"; others that logic concerns the "laws of thought." This paper presents a way to reconcile both views by building a bridge between truth-maker theory, à la Fine, and normative bilateralism, à la Restall and Ripley. The paper suggests a novel way of understanding consequence in truth-maker theory and shows that this allows us to identify a common structure shared by truth-maker theory and normative bilateralism. We can thus transfer ideas from normative bilateralism to truth-maker theory, such as non-transitive solutions to paradox, and vice versa, such as notions of factual equivalence and containment.

Keywords Strict-tolerant logic · Bilateralism · Truthmaker semantics · Factual equivalence · Substructural logic

1 Introduction

Inspired by Frege ([1], GG, xv-xvi), we may distinguish two broad conceptions of logic. According to the first, logic is concerned primarily with "laws of thought," i.e., with norms that govern acts or states in which concepts are used, such as judgments or assertions. According to the second conception, logic is concerned primarily with "laws of truth," i.e., with general structures or relations among the worldly items in virtue of which our judgments or assertions are true or false. There are many versions of these conceptions of logic. The aim of this paper is to show that two specific, influential versions of these conceptions, respectively, can be seen as two sides of one coin. The philosophical upshot is that we may not need to choose between them.

A particular, currently influential way to spell out the first conception of logic is normative bilateralism, which says that facts about what follows from what should

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be understood as facts about the normative status of collections of assertions and denials, so-called "positions" [2–7].[1] We write $[\Gamma : \Delta]$, with Γ and Δ being sets of sentences, for the position in which the sentences in Γ are asserted and those in Δ are denied. Positions that are ruled impermissible by a coherence norm are called "out-of-bounds." The other positions are "in-bounds." The central claim of normative bilateralism is that what it means for Δ to follow from Γ is that the position $[\Gamma : \Delta]$ is out-of-bounds. And for Δ to follow *logically* from Γ, written $\Gamma \models \Delta$, means that the position $[\Gamma : \Delta]$ is out-of-bounds in virtue of the logical form of the sentences in Γ and Δ.

One particular way to spell out the "laws of truth" conception of logic is truth-maker theory. According to truth-maker theory, there are parts or aspects of reality, called "states," that make some sentences true, i.e. verify them, and make some sentences false, i.e. falsify them [8, 9]. The verifying states (truth-makers) and falsifying states (falsity-makers) of a sentence are wholly relevant to the sentence. The state of it raining, e.g., is a truth-maker of the sentence "It is raining." But the state of it raining and it being cold is not a truth-maker of "It is raining" because it is not wholly relevant to the truth of the sentence. Consequence is then usually understood in terms of the truth- and falsity-makers of the involved sentences, e.g., as the relation that holds iff every truth-maker of (the conjunction of) the premise(s) is a truth-maker of the conclusion [8, 640-41].

The central idea of this paper is that an interesting equivalence emerges between truth-maker theory and normative bilateralism if we adopt an alternative conception of consequence in truth-maker theory, namely one that is the "metaphysical analogue" of normative bilateralism. I call it "truth-maker bilateralism." As I will show, this conception suggests a parallel structure between the "laws of truth" and the "laws of thought," as long as we understand these laws according to truth-maker theory and normative bilateralism respectively.[2] A notable strength of this bilateralist conception of consequence is that it enables us to translate philosophical ideas from truth-maker theory to normative bilateralism and vice versa. In the first direction, I show how three common constraints on possible states in truth-maker theory correspond to the traditional structural rules of cut, identity, and weakening in a sequent calculus. This yields, inter alia, results that are interesting from a technical perspective, such as a truth-maker semantics for the non-transitive logic ST. In the other direction, I show how versions of the ideas of equivalence, containment, and entailment in truth-maker theory can be given a unified interpretation in normative bilateralism. These results underline the central message of this paper: assuming the ideal case in which our

[1] I follow Restall and Ripley in formulating normative bilateralism in terms of speech acts. Some versions of the theory, however, use mental acts of acceptance and rejection instead. Because nothing below hinges on this distinction, I allow myself to use phrases like "laws of thought."

[2] The "laws of thought" conception is often pursued in a broadly proof-theoretic way, while the "laws of truth" conception is often pursued in a broadly model-theoretic way. Hence, it might be thought that soundness and completeness results already show that these conceptions are, in some sense, two sides of one coin. While soundness and completeness results will be important for me (see the technical Appendix), what I mean by "two sides of one coin" here is that the details of the theories are, in one important sense, equivalent, not just the resulting consequence relations.

norms of concept use correspond exactly to the modality of states, we can use a single logic to track consequence in both pragmatic and metaphysical terms.

The paper is structured as follows: In Section 2, I will suggest a way to understand consequence in terms of the impossibility of certain states; i.e., "truth-maker bilateralism."[3] Section 3 illustrates how this account allows us to use ideas developed in normative bilateralism within truth-maker theory; my main example is the non-transitive approach to semantic paradoxes. Section 4 illustrates the possibility of transferring ideas in the other direction; my main example is the notion of factual equivalence. Section 5 concludes.

2 Truth-Maker Bilateralism

This section articulates an account of consequence that combines aspects of normative bilateralism and truth-maker theory. I will first describe this idea in a bare-bones fashion. Then I will put some flesh on the bones and show how this yields classical propositional logic. Finally, I will bring out the common structure of truth-maker theory and normative bilateralism.

2.1 The Basic Idea

The central idea of this paper is an analogy between the basic elements of truth-maker theory and those of normative bilateralism: making-true is the worldly analogue of asserting, making-false is the worldly analogue of denying, and a state being impossible is the worldly analogue of a discursive position being out-of-bounds. Given this analogy, we can formulate a conception of consequence in truth-maker theory that follows the ideas of normative bilateralism: Δ follows from Γ iff every state that is (exactly) a fusion of a verifier for each sentence in Γ and a falsifier for each sentence in Δ is impossible. If we want to restrict ourselves to logical consequence, we consider only the states that are impossible in all models (which we define below). I call the resulting notion "truth-maker consequence" or "TM-validity" and denote it by $\models_{TM}$.

Truth-Maker Bilateralism (informal): $\Gamma \models_{TM} \Delta$ iff, in every model, any fusion of verifiers for each $\gamma \in \Gamma$ and falsifiers for each $\delta \in \Delta$ is an impossible state.

TM-validity underwrites many intuitions about consequence. It is, e.g., intuitive to think of consequence as a relation according to which the premises necessitate (the disjunction of) the conclusion(s). TM-validity cashes out this intuition as the claim that there is no possible state that includes exactly a truth-maker for each premise and a falsity-maker for each conclusion. This suggests, I think, that truth-maker bilateralism is an interesting candidate for one thing that we can reasonably mean by

[3] Fine's version of truth-maker theory already includes bilateral ideas. I will make this bilateral element central, hence the "-ism."

"consequence"; and I want to show in the rest of this paper that this conception of consequence is fruitful.

One of my background commitments is that philosophical accounts of what consequence is (or at least how it can reasonably be understood) and technical work on particular logics should inform and enrich each other.[4] It seems to me that stipulative, technical definitions of consequence relations are particularly interesting when they capture some plausible philosophical ideas about consequence; and philosophical ideas are more interesting if they can be captured in a framework that allows us to work out their implications in a rigorous way. My project here is located at this intersection, since it brings out the common structure in two stipulative definitions of consequence that aim to capture the "laws of thought" and "laws of truth" conceptions of logic, respectively.

2.2 Putting Formal Flesh on the Bones

In order to spell out truth-maker bilateralism, we must specify our models. I will limit myself throughout to propositional logic.[5] As we will see, there are different choices that lead to different consequence relations for TM-validity. Hence, $\models_{TM}$ is really a family of consequence relations. I will disambiguate where necessary by using appropriate labels. We start with what Fine [8, 647] calls a "modalized state space" and "fusion":

Definition 1 (Modalized state space) A modalized state space is a triple, $\langle S, S^{\Diamond}, \sqsubseteq\rangle$, such that S is a non-empty set of states, $S^{\Diamond} \subseteq S$ (intuitively: the possible states), and $\sqsubseteq$ is a partial order on S (intuitively: parthood), such that all subsets of S have a least upper bound.

We use $\square$ to denote the least upper bound of the empty set. This "null state" is part of every state.

Definition 2 (Fusion) The fusion of a set of states $T = \{t_1, t_2, t_3, ...\}$, written $t_1 \sqcup t_2 \sqcup t_3 ...$ or $\bigsqcup T$, is the least upper bound of T with respect to $\sqsubseteq$ [8, 646].[6]

Let $\mathfrak{L}$ be a language that results from adding $\neg$, $\vee$, and $\wedge$ to a countable stock of atomic sentences, $\mathfrak{L}_0$, in the usual way. Note that we don't include any modal

[4]See my [10] for more details on what I mean by an account of what consequence is, and how it interacts with technical work in logic.

[5]I do not foresee any particular problems with the extension of my results to first-order logic. However, there are well-known unresolved questions regarding the truth- and falsity-makers of universal generalizations ([11, sec I.7]; [12, chap 5]). I suspect that any plausible treatment of these issues can be reproduced within normative bilateralism.

[6]Here we can note a parallel to Girard's phase semantics. Girard [13, 18] defines a phase space as a commutative monoid together with a distinguished subset of the monoid set. Notice that $\langle S, \sqcup, \square\rangle$ is a commutative monoid, and $S^{\Diamond}$ (like $S \setminus S^{\Diamond}$) is a distinguished subset of the monoid set. This is worked out in much more detail in unpublished work by Daniel Kaplan (who is currently collaborating with me as a postdoc).

operators in our language; nevertheless the difference between possible and impossible states will be crucial, namely by figuring in the definition of consequence. An interpretation, $|A|$, of a sentence A—a bilateral proposition, in Fine's terms—is the pair consisting of the set of A's verifiers, written $|A|^+$, and the set of A's falsifiers, written $|A|^-$.

Definition 3 (Model) Given a language $\mathfrak{L}$, a model, $\mathcal{M}$, is a quadruple $\langle S, S^\Diamond, \sqsubseteq, |\cdot|\rangle$, where $\left\langle S, S^\Diamond, \sqsubseteq\right\rangle$ is a modalized state space and $|\cdot|$ is a interpretation function, such that $|A| = \left\langle |A|^+, |A|^-\right\rangle \in \mathcal{P}(S) \times \mathcal{P}(S)$.

We write $\mathcal{M}, s \Vdash A$ if a state s verifies a sentence A in model $\mathcal{M}$, and if no risk of confusion arises, simply $s \Vdash A$. Similarly, we write $s \dashV A$ to say that s is a falsifier of sentence A (in model $\mathcal{M}$). An interpretation must obey the following semantic clauses:

(atom+)	$s \Vdash p$ iff $s \in \lvert p\rvert^+$.
(atom−)	$s \dashV p$ iff $s \in \lvert p\rvert^-$.
(neg+)	$s \Vdash \neg B$ iff $s \dashV B$.
(neg−)	$s \dashV \neg B$ iff $s \Vdash B$.
(and+)	$s \Vdash B \wedge C$ iff $\exists u, t\ (u \Vdash B$ and $t \Vdash C$ and $s = u \sqcup t)$.
(and−)	$s \dashV B \wedge C$ iff
	$s \dashV B$ or $s \dashV C$ or $\exists u, t\ (u \dashV B$ and $t \dashV C$ and $s = u \sqcup t)$.
(or+)	$s \Vdash B \vee C$ iff
	$s \Vdash B$ or $s \Vdash C$ or $\exists u, t\ (u \Vdash B$ and $t \Vdash C$ and $s = u \sqcup t)$.
(or−)	$s \dashV B \vee C$ iff $\exists u, t\ (u \dashV B$ and $t \dashV C$ and $s = u \sqcup t)$.

It will prove convenient below to extend the definition of truth-makers and falsity-makers to sets of sentences as follows, where $\bigwedge \Gamma$ is the conjunction of exactly the members of Γ and $\bigvee \Delta$ is the disjunction of exactly the members of Δ:

Definition 4 (Truth- and falsity-makers of sets) $u \Vdash \Gamma$ iff $u \Vdash \bigwedge \Gamma$, unless $\{x : x \Vdash \bigwedge \Gamma\} = \emptyset$ in which case $\square$ and nothing else makes Γ true. And $t \dashV \Delta$ iff $t \dashV \bigvee \Delta$, unless $\{x : x \dashV \bigvee \Delta\} = \emptyset$ in which case $\square$ and nothing else makes Δ false.[7]

Given a countable infinity of states, there will be an uncountable infinity of propositions. Hence, we cannot express all propositions in a language. Since I want to ignore this complication, I will make the following assumption.[8]

[7] This definition has the consequence that the empty set is made true and false by the null state, $\square$. This may seem strange but it has technical advantages and is otherwise harmless. So I take this consequence on board.

[8] A lot of the work of this assumption can be done by using canonical models (see, [8, 647]). This works, e.g., for completeness proofs. However, the parallels I want to bring out in this paper can be seen more clearly with the assumption in place.

Assumption 1 (Expressibility of propositions) For any proposition $\langle V, F\rangle$, we can add sentences, $\Gamma \cup \Delta$, to our language such that every member of V contains exactly one truth-maker for each sentence in Γ and every member of F contains exactly one falsity-maker for each sentence in Δ.

So far, we have put no constraints on the possible states. Fine often imposes the following constraints, and they will become important below:

Downward-Closure: If $s \in S^{\diamond}$ and $t \sqsubseteq s$, then $t \in S^{\diamond}$.
Exclusivity: If $s \in |p|^{+}$ and $t \in |p|^{-}$, then $\forall u(s \sqcup t \sqcup u \notin S^{\diamond})$.[9]
Exhaustivity: $\forall u \in S^{\diamond}$, either $\exists s \in |p|^{+} (u \sqcup s \in S^{\diamond})$ or $\exists t \in |p|^{-} (u \sqcup t \in S^{\diamond})$.

Downward-Closure says that all parts of a possible state are possible. Exclusivity says that if you take any atomic[10] sentence, then if you fuse one of its verifiers with one of its falsifiers together with any other state, you always get an impossible state. And Exhaustivity says that if you have a possible state and an atomic sentence, then you can extend it to a possible state either by fusing it with a verifier of the sentence or by fusing it with a falsifier of the sentence. It will be one of the central results below that these three constraints correspond to the structural rules of weakening, identity, and cut, respectively.

So far, I have followed Fine's formulation of truth-maker theory [8, 9, 14–16]. However, we can now define truth-maker validity, which is my original contribution, in a formally precise way:

Truth-Maker Bilateralism: $\Gamma \models_{TM} \Delta$ iff, in every model, $s \notin S^{\diamond}$ for all $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\Vert \Delta$.

Given Definition 4, this says that an argument is TM-valid just in case every fusion of truth-makers for sentences in Γ and falsity-makers for the sentences in Δ is impossible. For the technical work below, this will serve as a stipulative definition. However, the philosophical interest of it lies in the fact that it captures the philosophical conception of consequence that I introduced in the previous subsection, which is in turn one way to spell out the "laws of truth" conception of logic. The central aim of this paper is to show how ideas can travel freely between truth-maker theory and normative bilateralism once we adopt truth-maker bilateralism. As a useful warm-up, I want to look at classical logic.

2.3 Classical Logic

There are known ways to recover classical logic within truth-maker theory [8, 665-668]. The way in which I will do it may seem complicated at first, but it will pay

[9] This formulation differs from Fine's in the quantification over further states u. In the presence of Downward-Closure, the two formulations are equivalent.

[10] Stipulating these constraints for atomic sentences suffices (given the semantic clauses) to enforce them for the whole language.

dividends later on. I will show that a sequent calculus for propositional classical logic is sound and complete with respect to $\models_{TM}$, given Downward-Closure, Exclusivity, and Exhaustivity.

Let $\succ$ be the sequent arrow, flanked by sets on both sides. I use uppercase Latin letters for sentences, lowercase Latin letters for atomic sentences, and uppercase Greek letters for sets of sentences. Double-lines indicate that the rule licenses a move in both directions, i.e., the usual top-to-bottom direction but also the derivation of any of the top sequents from the bottom sequent. Let's call the following sequent calculus "CL".

Structural Rules of CL:

$$\frac{}{\Gamma, p \succ p, \Delta}\,[\mathrm{ID}] \qquad \frac{\Gamma \succ \Delta}{\Theta, \Gamma \succ \Delta, \Sigma}\,[\mathrm{weakening}]$$

$$\frac{\Gamma \succ \Delta, p \qquad p, \Gamma \succ \Delta}{\Gamma \succ \Delta}\,[\mathrm{cut}]$$

Operational Rules of CL:

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\,[\mathrm{L}\wedge] \qquad \frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B \qquad \Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \wedge B}\,[\mathrm{R}\wedge]$$

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta \qquad \Gamma, A, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\mathrm{L}\vee] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\,[\mathrm{R}\vee]$$

$$\frac{\Gamma \succ \Delta, A}{\Gamma, \neg A \succ \Delta}\,[\mathrm{L}\neg] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \Delta, \neg A}\,[\mathrm{R}\neg]$$

A sequent is derivable, written $\Gamma \vdash_{CL} \Delta$, if there is a proof-tree of CL with the sequent as its root. CL is sound and complete with respect to classical propositional logic, denoted by $\models_{CL}$ (Appendix: Proposition 19).[11]

Some comments: [weakening], [cut], and the bottom-to-top directions of the double-line rules are redundant (Appendix: Propositions 16, 17, and 18). But they will be illuminating for us. Furthermore, the rules [R∧] and [L∨] will be unfamiliar to the reader because they include a third top-sequent in addition to the two standard top-sequents. In the presence of [weakening], however, this third top-sequent is redundant. The rules of CL correspond in a straightforward way to aspects of our truth-maker theory. Most importantly for the next section, the three structural rules correspond exactly to the three constraints on possible states.

Proposition 1 *The rules [ID], [weakening], and [cut] preserve TM-validity iff possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively (Appendix: Proposition 20).*

[11] I relegate as many technical details as possible to the Appendix, where I provide proofs of claims in the main text and lemmas. I will note this in the text in the way illustrated here.

The operational rules of our sequent calculus correspond to the semantic clauses in our truth-maker theory. The following definition allows me to formulate this succinctly in Lemma 1.

Definition 5 (Deeming impossible) A sequent $\Gamma \succ \Delta$ deems any state impossible that is a fusion of verifiers for everything in Γ and falsifiers for everything in Δ, i.e., any state $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\Vert \Delta$.

Lemma 1 *For every top-to-bottom application of an operational rule of* CL*, the set of states deemed impossible by the bottom-sequent is the union of the sets of states deemed impossible by the top-sequents (appendix: Lemma 2).*

The lemma implies that our operational rules cannot lead from TM-valid sequents to TM-invalid sequents, hence, our operational rules of CL are valid for $\models_{TM}$. Note that the lemma is independent of any constraints on possible states. The semantic clauses for the connectives suffice to ensure the result. Hence, we can use this result even if we change our constraints on possible states, which we will do in the next section.

Putting these results together suffices to show that $\models_{TM}$ coincides with classical propositional consequence (see the Appendix for details).

Proposition 2 *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then* $\Gamma \models_{TM} \Delta$ *iff* $\Gamma \vdash_{CL} \Delta$ *iff* $\Gamma \models_{CL} \Delta$ *(Appendix: Proposition 23).*

We have thus recovered classical logic in truth-maker theory. In order to see the significance of these considerations, it is important to note that the various parts of our truth-maker theory map onto our sequent calculus not just globally, but also in a local, piece-by-piece fashion. I will bring this out in the next subsection.

2.4 Two Sides of One Coin

We saw that we can give a sequent calculus presentation of a version of truth-maker validity that coincides with classical logic. In this subsection, I want to explain what this means for the relation between truth-maker bilateralism and normative bilateralism.

Normative bilateralism offers an intuitive interpretation of sequents and sequent rules. That the sequent $\Gamma \succ \Delta$ is valid, e.g., can be understood as the claim that the position $[\Gamma : \Delta]$ is out-of-bounds. And a sequent rule tells us that some positions are out-of-bounds if certain other positions are out-of-bounds. We can now give parallel interpretations of sequents and sequent rules in terms of truth-maker theory: a sequent rule tells us that some states are impossible if certain other states are impossible. Let me illustrate this by going through our sequent rules.

If we generalize the cut-rule to its usual (additive) formulation with arbitrary complex sentences, we can formulate the normative bilateralist (NB) and the truth-maker bilateralist (TM) interpretations of cut thus:

NB-cut: For any in-bounds position and any sentence, A, one can extend the position to an in-bounds position by either asserting or denying A.

TM-cut: For any possible state and any sentence, A, one can extend the state into a possible state by fusing it with either a verifier or a falsifier of A.

For a similarly generalized version of [ID], the two interpretations are the following:

NB-ID: Any position in which any sentence is asserted and also denied is out-of-bounds.

TM-ID: Any state that includes a verifier and also a falsifier for any sentence is impossible.

And here are the two interpretations of the weakening rule.

NB-weak: All positions that include an out-of-bounds position are themselves out-of-bounds.

TM-weak: All states that include an impossible state are themselves impossible.

For the left-rules, we can provide parallel interpretations in the following way:

NB-left: Left-rules specify the contributions that the assertions of complex sentences make to positions being out-of-bounds in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-left: Left-rules specify the contributions that the verifiers of complex sentences make to states being impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

And the parallel formulations for the right-rules are as follows:

NB-right: Right-rules specify the contributions that the denials of complex sentences make to positions being out-of-bounds in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-right: Right-rules specify the contributions that the falsifiers of complex sentences make to states being impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

If we map out-of-boundness to impossibility, assertions of sentences to their verifications, and denials of sentences to their falsifications, it is obvious that the two interpretations are isomorphic. That is, given the same sequent calculus, this mapping defines a theoremhood-preserving bijection between claims in normative bilateralism about positions being out-of-bounds and claims in truth-maker bilateralism about sets of states being impossible in all models. We have one formal structure with two interpretations—one metaphysical-alethic, the other pragmatic-normative. Hence, if normative bilateralism is our preferred way of spelling out the idea that logic concerns the laws of thought and truth-maker bilateralism is our preferred way of spelling out the idea that logic concerns the laws of truth, then we can see the two kinds of laws as sharing a common structure. In both frameworks, the consequence relation holds between two sets just in case certain states or positions are ruled out. The modality of this “ruling out” is normative in one case and alethic in the other case. What is the relation between these two flavors of modality in our two frameworks?

There are many kinds of normative defect of positions and many kinds of alethic impossibility of states. Plausibly, there are matching pairs in these two arrays. For example, some positions are out-of-bounds in the sense that they are conceptually incoherent, such as the position in which one asserts that something is a violin and denies that it is an artifact. There is a matching kind of conceptual impossibility, namely that of any state that combines a truth-maker for “This is a violin” and a falsity-maker for “This is an artifact.” Given that my official topic is logic, I am focusing on matching pairs of logical incoherence and logical impossibility, i.e., incoherence and impossibility in virtue of logical form. It is, e.g., logically incoherent to assert and deny the same sentence, and any state that combines a truth-maker and a falsity-maker of one and the same sentence is logically impossible.

For each such matching pair, it is plausible that the norms governing concept use should be such that they rule out-of-bounds precisely those positions that correspond to a state that is impossible in the matching sense. Applied to logic, ideally, our norms for deeming positions as logically incoherent should deem logically incoherent precisely those positions for which any fusion for verifiers for the assertions and falsifiers for the denials is logically impossible. This clarifies the sense in which the laws of thought and the laws of truth are two sides of one coin: on the one side, as the structure of norms governing concept use (the norms of taking-true), and on the other, as that of the impossibility of states (the laws of making-true). In the ideal case in which our norms of concept use are flawless, they correspond exactly to the impossibility of states. In such a case, we could track the norms governing what is (logically) out-of-bounds and the facts about which states are (logically) impossible by specifying a single consequence relation.

Now, what I said doesn’t exclude the possibility of one side of this coin being more fundamental. Those who think that logic is primarily concerned with the laws of truth will probably accord the metaphysical-alethic side priority, while those who think logic is primarily concerned with the laws of thought will do the opposite. My hope, however, is that the isomorphism I am pointing to will help us resist the apparent need to choose between these two options.

Here is one way to formulate this possibility, which I find attractive but is not mandatory for my purposes: States can plausibly be individuated by the sets of states with which they are (alethically) incompatible. For if two states are incompatible with all and only the same other states, then there is no possibility in which it makes any (modal) difference which of the two states obtains. Similarly, contents can plausibly be individuated by the sets of contents with which they are (normatively) incompatible. For if two contents are incompatible with all and only the same other contents, then there is no possible discourse in which it makes any (normative) difference which of the two contents occurs. Hence, if the laws of thought and the laws of truth match in the way suggested above, the content of an assertion/denial and the states that makes it true/false are individuated by playing identical roles with respect to other such contents and states, respectively. It seems to me that this gives us an attractive account of the sense in which contents correspond to reality. They play the same roles in their respective individuating structures. This is recognizably a version of the Aristotelian idea that the state of thinking something must share its (Aristotelian) form with what is thought, if we replace (Aristotelian) forms with our roles.

If that is correct, to study the structure that bilateralism and truth-maker theory share is to study these shared roles. Thinking of logic in this way strikes me as at least an interesting alternative to choosing proleptically between the two conceptions.[12]

More tangible attractions of what I have presented so far are, I hope, the following: (a) we can give an intuitive interpretation of sequent calculi within truth-maker theory; (b) we have a new notion of consequence for truth-maker theory that has proven interesting; and (c) this notion of consequence allows us to see the common structure in normative bilateralism and truth-maker theory.

Although these attractions are significant, stopping at classical logic would be unsatisfying. First, classical logic often feels like a straitjacket; since truth-maker theory, being hyperintensional, clearly has the capacity to free us from this straitjacket, it would be profitable to see how this plays out in truth-maker bilateralism. Second, given how simple classical logic is, an opponent might worry that it is not surprising that we can extract such a simple structure from our two frameworks. It would be much more impressive if the structural correspondence remained after parallel tweaks in both frameworks. That will be the topic of the next section.

3 Truth-Maker Substructurality à la Normative Bilateralism

We saw in the previous section that truth-maker bilateralism and normative bilateralism share a common structure. In the remainder of this paper, I want to show how deep and fruitful this commonality is. In this section, I show how we can take ideas about substructural logic from normative bilateralism and recast them within truth-maker theory. The results in this section will be of some independent interest, I hope, for efforts to capture substructural logics within truth-maker theory. Majer, Punčochář, and Sedlàr [18] present a truth-maker semantics for the Lambek calculus. And Jago [19] gives a truth-maker semantics for relevance logic. However, there is no known truth-maker semantics for non-transitive or non-reflexive logics, such as ST and TS respectively (see, [20]). In this section, I will present a truth-maker semantics for ST and TS, as well as truth-maker presentations of the structural logics LP and K3.

My approach to truth-maker semantics for substructural logics differs from extant approaches, which typically make no use of modal notions. Given truth-maker bilateralism, rejections of transitivity, reflexivity, and weakening correspond to rejections of Exhaustivity, Exclusivity, and Downward-Closure regarding possible states, respectively.[13] I will leave the question of whether rejecting one or more of these

[12]Readers familiar with Brandom's [17] "bimodal hylomorphic conceptual realism" may have realized that the current paper can be read as an elaboration of Brandom's ideas. However, I hope that my ideas are independently intelligible and plausible. Note that while Brandom is thinking of the shared content as matter, I am thinking of it as form.

[13]Rejecting permutation or contraction would require that we reject the commutativity or idempotence of fusion, which would in turn require that we not define fusion as a least upper bound. I will not pursue this avenue. If we think that the idempotence of fusion is essential to truth-maker theory but that contraction is not essential to normative bilateralism, then this indicates an interesting difference between our two frameworks and a sense in which their correspondence is not intrinsic or essential to them (thanks to a referee for pressing me on this point).

principles regarding possible states is philosophically defensible for another occasion. I merely want to bring out that the isomorphism I described in the previous section does not break down under such revisions; this is interesting and suggestive in its own right, because it allows us to see how one could argue for or against a structural principle such as [cut] independently of any particular logic, by appealing to parallel philosophical ideas from metaphysics and pragmatics.

3.1 Truth-Maker Semantics for ST

This subsection shows how truth-maker bilateralism allows us to translate the non-transitive approach to semantic paradoxes from normative bilateralism to truth-maker theory.

If we consider rejecting the principle NB-cut, this tells us what it means, given normative bilateralism, to reject the transitivity of consequence. Ripley [6, 7] has argued that normative bilateralists should do so in light of the semantic paradoxes.[14] To understand this non-transitive approach, let's add to our object language a canonical name $\bar{A}$ for every sentence A and a truth-predicate, *Tr*, for which we add the following sequent rules to CL.[15]

$$\frac{\Gamma, A \succ \Delta}{\Gamma, Tr(\bar{A}) \succ \Delta}\ [\mathrm{Lt}] \qquad\qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ Tr(\bar{A}), \Delta}\ [\mathrm{Rt}]$$

Let us allow for self-reference by allowing sentences that include their own names. Thus, we can formulate a liar sentence, $\neg Tr(\bar{\lambda})$, whose name is $\bar{\lambda}$ and that says of itself that it is not true. Note that λ is everywhere intersubstitutable with $\neg Tr(\bar{\lambda})$ *salva consequentia*.[16] Since $Tr(\bar{\lambda})$ is an atomic sentence, [ID] yields $Tr(\bar{\lambda}) \succ Tr(\bar{\lambda})$. Using [Lt], [Rt], our negation rules, and the intersubstitutability of $\neg Tr(\bar{\lambda})$ and λ, we can derive $\succ \lambda$ and $\lambda \succ$ [7]. Applying [cut] now yields the empty sequent. Something has to give, but it is notoriously difficult to motivate the rejection of any of the involved principles. Ripley argues that normative bilateralism offers a motivation for rejecting cut, for it is intuitively plausible that adding an assertion of the liar sentence or a denial of the liar sentence to an in-bounds position, $[\Gamma : \Delta]$, makes the position out-of-bounds, i.e., $[\Gamma : \Delta, \lambda]$ and $[\lambda, \Gamma : \Delta]$ are both out-of-bounds [7, 152]. In working out this idea, Ripley and others have formulated a non-transitive logic called ST that can accommodate a transparent truth-predicate and whose consequence relation includes every classically valid inference [21–23]. They show that

[14] Restall [5] disagrees with Ripley on this and holds that normative bilateralism motivates and underwrites the rule of cut. My sympathies lie with Ripley's view here, and I think the truth-maker version of Ripley's account of truth-theoretic paradoxes that I give here can lend support to Ripley's side of the disagreement. But this debate is beyond the scope of this paper. Here, I merely want to show that Ripley's position can be translated into truth-maker bilateralism.

[15] For simplicity, I leave out quantifiers, restricting us, in effect, to pure predicate logic. The truth-predicate and the name of the liar sentence are the only things we really need from the language of the predicate calculus.

[16] Here, I stipulate means of self-reference by fiat: I let λ and $\neg Tr(\bar{\lambda})$ simply be identical. This allows us to avoid the complications of adding self-reference via Gödel numbering or the like. See Ripley [21, 355] for more details.

cut is admissible in ST, even when cutting on sentences involving the truth-predicate, as long as the derivations of the premise-sequents don't use the truth-rules [7, 148].

We can formulate ST in truth-maker bilateralism. All we need to do is to drop the requirement of transitivity and add rules that govern the truth-predicate, which I will now assume to be part of our language. To do so, we tweak our sequent calculus and our truth-maker theory in parallel ways. First, we tweak our sequent calculus:

Definition 6 (Sequent Calculus $\mathsf{CL}^{+\backslash[cut]}$) Let $\mathsf{CL}^{+\backslash[cut]}$ be the sequent calculus that is like CL except that it includes the rules [Lt] and [Rt] and doesn't include the rule [cut]. We say that $\Gamma \mid\!\!\frac{}{\mathsf{CL}^{+\backslash[cut]}} \Delta$ iff the sequent $\Gamma \succ \Delta$ is derivable in $\mathsf{CL}^{+\backslash[cut]}$.

The provable sequents of this calculus coincide with those of (the propositional fragment of) ST, i.e., $\Gamma \mid\!\!\frac{}{\mathsf{CL}^{+\backslash[cut]}} \Delta$ iff $\Gamma \mid\!\!\frac{}{\mathsf{ST}} \Delta$ (Appendix: Proposition 24). We now make the analogous changes in our truth-maker theory. First, we need the following clauses for our truth-predicate:

(tr+) $\quad s \Vdash Tr(\bar{A})$ iff $s \Vdash A$

(tr−) $\quad s \dashv\!\mid\!\mid Tr(\bar{A})$ iff $s \dashv\!\mid\!\mid A$

Since we defined λ as $\neg Tr(\bar{\lambda})$, they have the same truth-makers and falsity-makers. As is suggested by our parallel, this together with (tr+) and (tr−) would trivialize $\models_{TM}$ in the presence of our three constraints on possible states (Appendix: Proposition 25). The non-transitive solution is, of course, to drop Exhaustivity as a constraint on possible states.

Definition 7 (ST-TM-Validity) Let $\models^{ST}_{TM}$ be the consequence relation just like $\models_{TM}$ except that we include models that violate Exhaustivity and the language has a truth-predicate whose interpretation obeys the clauses (tr+) and (tr−).

For normative bilateralism, the intuitive idea behind the rejection of cut was that the liar sentence can be neither coherently asserted nor coherently denied. When we translate this into truth-maker bilateralism, the result is that any state that includes a verifier or a falsifier of the liar sentence is impossible, yielding a violation of Exhaustivity. Since the world is a possible state, we can express the idea by saying that the world cannot contain anything that makes the liar sentence either true or false. Just as we should neither assert nor deny the liar sentence, so the world can neither verify nor falsify it. The non-transitive versions of normative bilateralism and truth-maker bilateralism aren't merely similar; they are structurally identical, in the sense that they yield the same logic.

Proposition 3 $\Gamma \mid\!\!\frac{}{\mathsf{CL}^{+\backslash[cut]}} \Delta$ *iff* $\Gamma \models^{ST}_{TM} \Delta$ *(Appendix: Proposition 26).*

So $\models^{ST}_{TM}$ offers a truth-maker semantics for ST. Thus, we have a truth-maker semantics for the currently most popular non-transitive logic.

3.2 Recasting Logics Related to ST in Truth-Maker Theory

The discovery that ST includes every classically valid inference and also a transparent truth-predicate has sparked fruitful research on ST's relations to other logics, especially LP and K3 (see, [20]). In this subsection, I illustrate how truth-maker bilateralism allows us to use this research to give independently interesting characterizations of LP, K3, and the non-reflexive logic TS. A key result of the research just mentioned is that LP is what is called the "external logic" of several sequent-calculus presentations of ST [20, 24, 25]. In our context, this means the following:

Proposition 4 $\Gamma \models_{LP} \Delta$ *iff* $\succ\Delta$ *is provable in* $\mathsf{CL}^{+\backslash[cut]}$ *if we add* $\succ\gamma$ for all $\gamma \in \Gamma$ *as axioms (Appendix: Proposition 27).*

We can now translate this fact into truth-maker theory.

Proposition 5 $\Gamma \models_{LP} \Delta$ *iff, in all models, no fusion of falsifiers of everything in* Δ *is possible if no falsifier of anything in* Γ *is possible, i.e.,* $\forall d \in |\bigvee \Delta|^- \ (d \notin S^\Diamond)$ *in all* $\models^{ST}_{TM}$ *models in which* $\forall\gamma \in \Gamma\ (\forall g \in |\gamma|^- (g \notin S^\Diamond))$ *(Appendix: Proposition 28).*

Thus LP emerges as the logic of falsity-makers, in the sense that it preserves the impossibility of falsity-makers from the premises (individually) to the conclusions (jointly). It is well-known that K3 and LP are dual logics, i.e., $\Gamma \models_{K3} \Delta$ iff $\neg\Delta \models_{LP} \neg\Gamma$, where $\neg X$ is the set of formulas that includes exactly the negations of the members of X (see, [20]). It follows from this that:

Proposition 6 $\Gamma \models_{K3} \Delta$ *iff* $\Gamma\succ$ *is provable in* $\mathsf{CL}^{+\backslash[cut]}$ *if we add* $\delta\succ$ for all $\delta \in \Delta$ *as axioms (Appendix: Proposition 29).*

Hence, we can now give a characterization of K3 as concerning truth-makers.

Proposition 7 $\Gamma \models_{K3} \Delta$ *iff, in all models, no fusion of verifiers of everything in* Γ *is possible if no verifier of anything in* Δ *is possible, i.e.,* $\forall g \in |\bigwedge \Gamma|^+ \ (g \notin S^\Diamond)$ *in all* $\models^{ST}_{TM}$ *models in which* $\forall\delta \in \Delta\ (\forall d \in |\delta|^+ (d \notin S^\Diamond))$ *(Appendix: Proposition 30).*

Put differently, $\Gamma \models_{K3} \Delta$ says that if all the premises together are possibly true, then some conclusion is also possibly true. In addition to the structural logics K3 and LP, recent research on ST often discusses a non-reflexive logic called TS.[17] In order

[17] The logic TS has been endorsed (tentatively) as a response to the semantic paradoxes by French [26]. There is a well-known definition of TS in a three-valued semantics [20, see e.g.]. French [26, 118] uses a proof-theoretic characterization, which isn't exactly mine. I will use the label TS for the sequent calculus that results from ST by removing [ID] while adding [cut]. I take this to be in agreement with what French means by TS; none of the differences matter for anything at issue here.

to recover TS, we can now again tweak our sequent calculus and truth-maker-bilateralist consequence relation in corresponding ways by dropping [ID] and Exclusivity respectively.

Definition 8 (Sequent Calculus $\mathsf{CL}^{\setminus[ID]}$) Let $\mathsf{CL}^{\setminus[ID]}$ be the sequent calculus that is like CL except that it doesn't include the rule [ID].

Definition 9 (TS-TM-Validity) Let $\models^{TS}_{TM}$ be the consequence relation just like $\models_{TM}$ except that we include models that violate Exclusivity.

It is easy to see—but hardly impressive—that all three relations are equivalent and empty (Appendix: Proposition 31). However, these systems are not just equivalent at the (empty) inferential level; they also coincide at the meta-inferential level. In other words, if we close a set of implications under the principles of these logics, we get the same results in all three cases.

Proposition 8 *Let $\mathfrak{I}$ be a set of pairs, $\langle \Gamma, \Delta \rangle$, where Γ and Δ are sets of sentences. Regardless of whether we close the relation $\mathfrak{I}$ under the rules of $\mathsf{CL}^{\setminus[ID]}$, of TS, or of $\models^{TS}_{TM}$, the resulting consequence relation is the same (Appendix: Proposition 32).*

In this subsection, I showed that, given truth-maker bilateralism, not only can we translate ST into truth-maker theory, but this translation also extends to related logics, in particular LP, $\mathsf{K3}$, and TS. These results illustrate that normative bilateralism and truth-maker bilateralism coincide not only in the classical case, but also in their relations to a wealth of well-known logics.[18]

4 Truth-Maker Contents in Normative Bilateralism

In the previous section, I showed how truth-maker bilateralism allows truth-maker theorists to adopt and further develop ideas stemming from normative bilateralism. In this section, I want to show how truth-maker bilateralism also allows ideas to flow in the other direction. I will start by considering the version of analytic equivalence developed by Correia [29], who calls it "factual equivalence." I will then turn to (variants of) Fine's notions of containment, entailment, and subject matter. The general takeaway is that various notions of consequence and equivalence in truth-maker theory can be recast in normative bilateralism in terms of the inferential roles of sentences, i.e., the intersubstitutability of sentences as premises or conclusions.

[18] It is worth noting that the truth-maker semantics for ST and TS is more flexible than the usual three-valued semantics. We can, e.g., construct a truth-maker semantics for any non-monotonic logic that can be obtained by closing an arbitrary (non-monotonic) set of atomic sequents under the operational rules of CL. For reasons of space, I must leave the details for another occasion. For more on these kinds of non-monotonic consequence relations see my [27, 28].

4.1 Factual Equivalence

Correia is interested in determining when two sentences, A and B, describe the same facts or states in virtue of their (propositional) logical form, written $A \approx B$. I will show how this logic can be recovered from our sequent calculus CL, thus allowing us to interpret it using normative bilateralism.

I use "Correia's logic" for the logic presented in [29]. It is a proper fragment of Angell's [30, 31] logic of analytic entailment, which Fine [16] advocates as a logic of content. Correia's logic differs from Angell's logic in that it doesn't validate the distributive principle according to which $A \vee (B \wedge C)$ is equivalent to $(A \vee B) \wedge (A \vee C)$. Correia provides philosophical arguments for accepting failures of this principle in truth-maker theory. I agree with Correia, but I will not engage this debate.[19]

Correia provides two semantic characterizations and a Hilbert-style axiomatisation of his logic. One of these semantic characterizations has an illuminating relation to the sequent rules of CL: according to Correia's "supersentential" semantics, a supersentence is a pair of set of sentences, written $\Delta \mid \Gamma$. Some such supersentences are fitting descriptions of states, where Correia takes the notion of a fitting description to be primitive. Correia then connects this notion with Fine's notions as follows: a supersentence fittingly describes a state, s, iff (i) each sentence in Δ is (exactly) falsified by some parts of s, (ii) each sentence in Γ is (exactly) verified by some parts of s, and (iii) s is the fusion of all the parts which falsify members of Δ or verify members of Γ [29, 110]. Hence, the states that are fittingly described by the supersentence $\Delta \mid \Gamma$, according to Correia, are precisely those states that, in my terminology, are deemed impossible by the sequent $\Gamma \succ \Delta$. Thus translated, it is easy to verify that Correia's supersentences obey the operational rules of CL, i.e., the states fittingly described by the conclusion of a top-to-bottom rule application is the union of the states fittingly described by the premises. While I don't make use of this machinery here, the propositions below can easily be shown using Correia's supersentential semantics.

In this section, I will use the semantics from Section 2 and Correia's Hilbert-style system, which includes the following axiom (using Correia's label).

A10 $\quad A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$

We get Angell's first-degree system for analytic equivalence by adding the distribution principle that Correia rejects as another axiom, namely:

A11 $\quad A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

[19] Formally speaking, the fact that I agree with Correia on this point means I will not insist on propositions being what Fine calls "convex," i.e., I will not insist that every state that has a verifier of the proposition as a part and that is itself part of a verifier of the proposition is also a verifier of the proposition. This constraint is what allows Fine to adopt Angell's logic rather than Correia's logic. Since it is not obvious how the distribution law can be added in normative bilateralism when pursuing the strategy of this section, we may here see another way in which the correspondence between truth-maker theory and normative bilateralism could be broken (see Footnote 13).

If we add A11 but not A10, we get what Correia calls his "dual system."

Correia proves of the system with just A10 (which I call Correia's logic) that it is sound and complete with respect to equivalence in truth-maker theory in the following sense:

Fact 1 $A \approx B$ is a theorem of Correia's logic iff A and B have the same truth-makers in every truth-maker model [29].

Since truth-makers and falsity-makers vary independently of $S^{\Diamond}$ across models, Correia's result holds independently of the constraints we put on possible states. Hence, our operational sequent rules should suffice to capture Correia's factual equivalences. Indeed, they do so in the following way:

Proposition 9 *$A \approx B$ is a theorem of Correia's logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B \succ \Delta$ and vice versa are admissible (Appendix: Proposition 35).*

This means that two sentences have the same truth-makers in all models just in case, in virtue of their logical form, they play the same role as premises in truth-maker consequence, i.e., they are inter-substitutable as premises *salva consequentia*. Translating into normative bilateralism, we can say that two sentences are factually equivalent in virtue of their logical form just in case their logical form ensures that assertions of the two sentences always make the same contribution to the out-of-boundness of positions. Correia [29, 117] shows that $A \approx B$ is provable in his dual system iff $\neg A \approx \neg B$ is provable in his original system. Given the negation rules of CL, $\Gamma, \neg A \succ \Delta$ is derivable iff $\Gamma \succ A, \Delta$ is derivable, and $\Gamma, \neg B \succ \Delta$ is derivable iff $\Gamma \succ B, \Delta$ is derivable. This implies:

Proposition 10 *$A \approx B$ is a theorem of Correia's dual logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible iff A and B have the same falsity-makers in all models (Appendix: Proposition 36).*

Hence, factual equivalence in Correia's dual system holds between sentences just in case the sentences play, in virtue of their logical form, the same inferential role as conclusions. In the language of normative bilateralism, what Correia's dual system captures is when the denials of two sentences make the same contribution, in virtue of their logical form, to the out-of-boundness of positions.

4.2 Containment, Entailment, and Subject Matter

We can now also recover a variant of Fine's notion of containment, modulo the distribution principle A11 above.[20] Fine defines containment as follows [8, 640-41].

[20] As already intimated, my notions differ from Fine's insofar as he requires propositions to be convex.

Definition 10 (Containment) A contains B iff (i) every verifier of A includes as a part a verifier of B and (ii) every verifier of B is included as a part in a verifier of A.

Since containment is defined purely in terms of verifiers, it should be recoverable within bilateralism as concerning the left side of sequents. Indeed, we can recover containment in virtue of logical form as follows.

Proposition 11 *A contains B, in virtue of logical form, iff, for some Θ, the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible (Appendix: Proposition 37).*

Thus the containment holds just in case A and $\{B\} \cup \Theta$ play the same inferential roles as premises, so that this inferential role of A as a premise has a part that is the inferential role of B as a premise. In the language of normative bilateralism, what it means for A to contain B, in virtue of logical form, is that there is a set of sentences, Θ, such that an assertion of A and the joint assertions of everything in $\{B\} \cup \Theta$ always have the same effect on the coherence of one's position, in virtue of logical form.

Besides containment, Fine often uses the notion of entailment, which he defines as follows [8, 640-41].

Definition 11 (Entailment) A entails B iff every verifier of A is a verifier of B.

The following result allows us to recast entailment in virtue of logical form in terms of our sequent rules.

Proposition 12 *A entails B, in virtue of logical form, iff the operational rules of* CL *suffice to show that the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible (Appendix: Proposition 38).*

Thus, in the language of normative bilateralism, that A entails B in virtue of logical form says that if a position is made incoherent by asserting B, then the position is also made incoherent by asserting A, in virtue of logical form. Put differently, the entailment says that if you can coherently assert A, then you can coherently assert B. Let me end by giving a characterization of what Fine calls the "subject matter" of a proposition [15, 697].

Definition 12 (Subject-matter) The subject-matter of a bilateral proposition is a pair in which the first element is the fusion of all of its verifiers and the second element is the fusion of all of its falsifiers.

If we do a proof-search on $A\succ$, we get a set of atomic sequents such that the union of all the states that any of these sequents deems impossible are exactly the truth-makers of A (Appendix: Proposition 33). And since the falsity-makers of A are the truth-makers of $\neg A$, a proof-search on $\neg A\succ$ and hence, on $\succ A$ yields a set of atomic sequents that together deem exactly the falsity-makers of A impossible. So, it is obvious that we can characterize subject-matter as follows:

Proposition 13 *Given the proof-trees that result from proof-searches on $A\succ$ and $\succ A$, the subject-matter of the proposition that A expresses is the pair, $\langle \boldsymbol{a}, \boldsymbol{a'} \rangle$, where $\boldsymbol{a}$ is the fusion of states deemed impossible by the leaves of the proof-tree for $A\succ$, and $\boldsymbol{a}'$ is this fusion for $\succ A$.*

In normative bilateralism, the leaves of a proof tree for $A\succ$ give us the atomic positions (positions in which only atomic sentences are asserted or denied) such that if one asserts just A—free-standingly, as the sole element in one's position—then one is out-of-bounds iff all of these atomic positions are out-of-bounds. And similarly for a proof tree for $\succ A$ and a free-standing denial of A. Thus we can think of the subject-matter of a proposition, expressed by A, as the worldly analogue of the atomic positions that are equivalent (in their coherence status) with free-standing assertions and denials of A. The two elements of the subject matter are, as it were, the worldly analogue of the projections of the assertion and denial of A, respectively, onto atomic positions.

To sum up, I have shown how we can recast ideas that were developed in truth-maker theory within normative bilateralism. It is interesting to note that the fine-grained logical relations of equivalence, containment, and entailment emerge as relations among inferential roles, i.e., relations among sentences that can be replaced for one another as premises (or conclusions, for Correia's dual system) *salva consequentia*. This illustrates how the theory of content and meaning provided by truth-maker theory can be recast in inferentialist terms (in keeping with the fact that normative bilateralists usually identify as inferentialists).

5 Conclusion

I have shown that truth-maker bilateralism and normative bilateralism share a common structure. There are general principles governing which states or positions are ruled out (metaphysically or normatively, respectively), which correspond to the reflexivity, transitivity, and monotonicity of the consequence relation. And there are rules for the logical connectives that specify the contributions a logically complex sentence makes to the status of a state or position in terms of such contributions by the sentence's constituents. This correspondence allows ideas to flow between truth-maker theory and normative bilateralism, as we have seen for the non-transitive approach to semantic paradoxes and the notion of factual equivalence.

Future research might explore just how far the parallel between truth-maker bilateralism and normative bilateralism can be extended. Normative bilateralists have, for instance, used hypersequent calculi to formulate accounts of modal logics [32, 33] and n-sided sequent calculi to formulate accounts of n-valued logics [34, 35]. I hypothesize that these accounts can be translated into truth-maker bilateralism in a straightforward way. Regarding the other direction, truth-maker theory has been used to give accounts of imperatives [36, 37], the is-ought gap [38], verisimilitude [39], and counterfactuals [40]. It is worth investigating how much of these accounts could be recovered within truth-maker bilateralism and translated into normative bilateralism.

The structural equivalence of truth-maker bilateralism and normative bilateralism promises a new philosophical approach to logic. Instead of thinking of logic as concerned with either the laws of thought or the laws of truth, we can think of it as concerned with the shared structure of these kinds of laws. Facts about this structure are in a fruitful way systematically ambiguous: they can be interpreted as concerning the norms governing assertions and denials or concerning the states that make truth-bearers true or false. In this way, we are not forced to choose one of the two conceptions of logic with which we started; rather, we can adopt as a desideratum on our logical theories that they illuminate simultaneously the nature and structure of both the worldly states we talk about and the discursive norms we must follow in order to talk about them.

Appendix A: Proofs and Technical Details

A.1 Classical Logic

Definition 13 (Proof-search) A root-first proof-search produces a proof-tree from a sequent $\Theta \succ \Sigma$, which is the root of the tree, by recursively applying the following procedure until the process terminates: (i) If $\Gamma \succ \Delta$ is the leaf of a branch of the tree at the current stage and all the sentences in Γ and Δ are atomic, then the branch remains unchanged. (ii) Otherwise, we look for the first complex sentence in $\Gamma \succ \Delta$ (starting on the left, ordering the sentences in Γ and Δ alphabetically) and build the branch up from that leaf by using the appropriate rule of CL. For example, we apply the top-to-bottom version of [L$\vee$] (moving upwards in the tree) if the left-most complex sentence in our sequent is a disjunction, etc. (Although we work with sets (and so contraction is built in), we represent the sets in our sequents with the number of copies of sentences that we get by applying this procedure to the given representation of the root, thus treating our sets (in how we represent them) like multi-sets.)

Someone might worry that this definition will yield different results for representations of the root sequent that differ in the numbers of copies of sentences. In fact, however, this doesn't happen.

Proposition 14 *Proof-searches on* $\Gamma, A \succ \Delta$ *and* $\Gamma, A, A \succ \Delta$ *yield the same results, and the same holds for proof-searches on* $\Gamma \succ A, \Delta$ *and* $\Gamma \succ A, A, \Delta$

Proof By induction on the complexity of A. □

Proposition 15 *Proof-searches terminate, and their results are the same if we change the order of the sentences in* Γ *and* Δ.

Proof Proof-searches terminate because the root contains finitely many logical connectives, and the children of a node always contain one fewer connective than the parent node. To show that the order doesn't matter, it suffices to show that for each pair of rules, the order in which they are applied doesn't matter. If we have, e.g., $\neg A, B \vee C, \Gamma \succ \Delta$, applying our procedure to the first two sentences yields: $B, \Gamma \succ \Delta, A$ and $C, \Gamma \succ \Delta, A$ and $B, C, \Gamma \succ \Delta, A$. This result is the same whether we use [L¬] first and then [L∨] or the other way around. The same holds for all pairs of rules. □

Proposition 16 *In* CL*, [weakening] is redundant.*

Proof We can add the desired additional context to every application of [ID]. □

Proposition 17 *In* CL*, the bottom-to-top operational rules can be eliminated, i.e., omitting these rules does not change which sequents are derivable.*

Proof We argue by induction on proof-height, and look at each bottom-to-top rule in turn. Since [weakening] can be eliminated, it suffices to look at proof-trees without [weakening]. I will give the proof for [L∧]; the other cases are analogous. Suppose we have a derivation of $\Gamma, A \wedge B \succ \Delta$. We must show that $\Gamma, A, B \succ \Delta$ is derivable. If $\Gamma, A \wedge B \succ \Delta$ was derived via [L∧], we're done. For all the other rules by which $\Gamma, A \wedge B \succ \Delta$ may come, $A \wedge B$ must have been in the left context of the rule-application. We can apply our induction hypothesis and replace the conjunction with the two conjuncts. We then get $\Gamma, A, B \succ \Delta$ by applying the rule by which $\Gamma, A \wedge B \succ \Delta$ was derived in our initial proof-tree. □

Proposition 18 *In* CL*, [cut] can be eliminated.*

Proof From the proof of Proposition 26 below, it is easy to see that CL without [cut] is equivalent to the sequent calculus of ST without the truth-rules, and it is well-known that [cut] is admissible in that sequent calculus. Hence, [cut] is admissible in CL without [cut]. □

Proposition 19 CL *is sound and complete with respect to classical propositional logic, i.e.,* $\Gamma \vdash_{CL} \Delta$ *iff* $\Gamma \models_{CL} \Delta$.

Proof For soundness, it suffices to note that every classical truth-assignment satisfies (i.e., is not a counter-model to) any instance of [ID] and that all rules of CL preserve that property. For completeness, suppose that $\Gamma \succ \Delta$ cannot be derived. Hence, a proof-search for $\Gamma \succ \Delta$ yields at least one atomic sequent, $\Gamma_0 \succ \Delta_0$, such that $\Gamma_0 \cap \Delta_0 = \emptyset$. So, there is a counterexample to $\Gamma_0 \succ \Delta_0$. Any counterexample to $\Gamma_0 \succ \Delta_0$ is also a counterexample to $\Gamma \succ \Delta$. By the contrapositive of [cut], for any atomic sentence, p, either $\Gamma_0 \succ \Delta_0, p$ or $p, \Gamma_0 \succ \Delta_0$ is not derivable. If $\Gamma_0 \succ \Delta_0, p$ is not derivable, we make p false; otherwise $p, \Gamma_0 \succ \Delta_0$ is not derivable, and we make p true. In this way, we can extend our counterexample by assigning truth-values to all atomic sentences. Hence, $\Gamma \not\models_{CL} \Delta$. □

Proposition 20 *The rules [ID], [weakening], and [cut] are valid for* $\models_{TM}$ *iff possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively.*

Proof Downward-Closure and [weakening]: Downward-Closure says that if $s \in S^{\Diamond}$ and $t \sqsubseteq s$, then $t \in S^{\Diamond}$. Now, if $\Gamma \models_{TM} \Delta$, then any fusion of verifiers of everything in Γ and falsifiers of everything in Δ is impossible. By Downward-Closure, all states that include any such fusion as a part are also impossible. Hence, $\Theta, \Gamma \models_{TM} \Delta, \Sigma$. For the other direction, suppose that [weakening] is valid and that $s \in S^{\Diamond}$ and $t \sqsubseteq s$. In accordance with Assumption 1, let Γ, Θ, Δ, and Σ be such that s is the unique state that is a fusion of verifiers for everything in $\Gamma \cup \Theta$ and falsifiers for everything in $\Delta \cup \Sigma$. Since $s \in S^{\Diamond}$, we know that $\Theta, \Gamma \not\models_{TM} \Delta, \Sigma$. If [weakening] is valid for $\models_{TM}$, it follows that $\Gamma \not\models_{TM} \Delta$. Since $t \sqsubseteq s$, without loss of generality, let t be the state that results from s by omitting the verifiers for Θ and the falsifiers for Σ. Then t is the unique state that is a fusion of verifiers for everything in Γ and falsifiers for everything in Δ. Hence, $t \in S^{\Diamond}$.

Exclusivity and [ID]: Exclusivity says that if $s \in |p|^+$ and $t \in |p|^-$, then $\forall u(s \sqcup t \sqcup u \notin S^{\Diamond})$. So, $\Gamma, p \models_{TM} p, \Delta$. For the other direction, suppose [ID] is valid and let $s \in |p|^+$ and $t \in |p|^-$. By [ID], for any Γ and Δ, we have $\Gamma, p \models_{TM} p, \Delta$. So every state that includes a truth-maker and a falsity-maker of p is impossible, i.e., $\forall u(s \sqcup t \sqcup u \notin S^{\Diamond})$.

Exhaustivity and [cut]: Suppose that $\Gamma \not\models_{TM} \Delta$ and let u be a state witnessing this fact, i.e., a state that is a fusion of verifiers of everything in Γ and falsifiers of everything in Δ such that $u \in S^{\Diamond}$. By Exhaustivity, $\exists s \in |p|^+(u \sqcup s \in S^{\Diamond})$ or $\exists t \in |p|^-(u \sqcup t \in S^{\Diamond})$. Therefore, either $p, \Gamma \not\models_{TM} \Delta$ or $\Gamma \not\models_{TM} \Delta, p$. But that is just what is required for the contrapositive of [cut]. For the other direction, suppose that [cut] is valid for $\models_{TM}$. Let u be possible; and, in accordance with Assumption 1, let Γ be a set such that u is the unique state that is a fusion of verifiers for everything in Γ. Hence, $\Gamma \not\models_{TM} \emptyset$. By the validity of [cut], either $p, \Gamma \not\models_{TM} \emptyset$ or $\Gamma \not\models_{TM} p$. Hence, either $\exists s \in |p|^+(u \sqcup s \in S^{\Diamond})$ or $\exists t \in |p|^-(u \sqcup t \in S^{\Diamond})$. □

Lemma 2 *For every top-to-bottom application of an operational rule of* CL*, the set of states deemed impossible by the bottom-sequent is the union of the states deemed impossible by the top-sequents.*

Proof I do the case for conjunction; the proofs for negation and disjunction are analogous. For [L$\wedge$]: Note that by our semantic clauses $|A \wedge B|^+ = \{s : \exists a \in |A|^+ \exists b \in |B|^+(s = a \sqcup b)\}$. Hence, for any Γ and Δ, we have $\{g \sqcup d \sqcup a \sqcup b : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $a \in |A|^+$ and $b \in |B|^+\} = \{g \sqcup d \sqcup s : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $s \in |A \wedge B|^+\}$.

Similarly for [R$\wedge$], note that $|A \wedge B|^- = |A|^- \cup |B|^- \cup \{s : \exists a \in |A|^- \exists b \in |B|^-(s = a \sqcup b)\}$. Therefore, $\{g \sqcup d \sqcup s : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $s \in |A \wedge B|^-\} = \{g \sqcup d \sqcup a : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $a \in |A|^-\}$ $\cup \{g \sqcup d \sqcup b : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $b \in |B|^-\} \cup \{g \sqcup d \sqcup c : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $c \in \{x : \exists a \in |A|^- \exists b \in |B|^-(x = a \sqcup b)\}\}$. □

Proposition 21 *All operational rules of* CL *are valid for* $\models_{TM}$.

Proof Immediate from Lemma 2. □

Proposition 22 *If a model is a counterexample to a top-sequent of a top-to-bottom application of an operational rule of* CL*, then the model is also a counterexample to the bottom-sequent.*

Proof By Lemma 2, if a state deemed impossible by a top-sequent is possible in $\mathcal{M}$, then a state deemed impossible by the bottom sequent is possible in $\mathcal{M}$. □

Proposition 23 *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then* $\Gamma \models_{TM} \Delta$ *iff* $\Gamma \vdash_{CL} \Delta$ *iff* $\Gamma \models_{CL} \Delta$.

Proof We show that $\Gamma \vdash_{CL} \Delta$ is sound and complete with respect to $\models_{TM}$ and to $\models_{CL}$. By Proposition 19, CL is sound and complete with respect to $\models_{CL}$. For $\models_{TM}$, we know soundness from Propositions 20 and 21. For completeness, suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a proof-search for $\Gamma \succ \Delta$ yields an atomic sequent, $\Gamma_0 \succ \Delta_0$, where $\Gamma_0 \cap \Delta_0 = \emptyset$. Let $\mathcal{M}$ be a model in which $s \in S^{\Diamond}$ and $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \mathrel{\dashv\!\mid} \Delta$. This is a counterexample to $\Gamma_0 \succ \Delta_0$. By Proposition 22, it is also a counterexample to $\Gamma \succ \Delta$. Since $\Gamma_0 \cap \Delta_0 = \emptyset$, such a model isn't ruled out by Exclusivity, which is the only principle that could rule out such a model. So, $\mathcal{M}$ shows that $\Gamma \not\models_{TM} \Delta$. □

A.2 Relation of $\mathsf{CL}^{\backslash[\mathbf{cut}]}$ to ST, LP, K3, and TS

I will use a slightly adjusted version of the propositional fragment of Ripley's [7] sequent calculus presentation of ST, namely the following:

Structural Rules of ST:
$$\frac{}{p \succ p}\,[\text{ID-ST}] \qquad \frac{\Gamma \succ \Delta}{\Theta, \Gamma \succ \Delta, \Sigma}\,[\text{weakening-ST}]$$

Operational Rules of ST:

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\text{L}\vee\text{-ST}] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\,[\text{R}\vee\text{-ST}]$$

$$\frac{\Gamma \succ \Delta, A}{\Gamma, \neg A \succ \Delta}\,[\text{L}\neg\text{-ST}] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \Delta, \neg A}\,[\text{R}\neg\text{-ST}]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, Tr(\bar{A}) \succ \Delta}\,[\text{Lt-ST}] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ Tr(\bar{A}), \Delta}\,[\text{Rt-ST}]$$

We write $\Gamma \vdash_{\mathsf{ST}} \Delta$ iff the sequent $\Gamma \succ \Delta$ is derivable in ST. Ripley [7] uses single-line rules and an additive right-rule for disjunction, and he includes the material conditional. Given [weakening-ST], the definability of the conditional as $A \supset B =_{\text{def.}} \neg A \vee B$, and the admissibility of the bottom-to-top rules in the single-line ST calculus, these differences don't change which sequents are provable.[21] Ripley treats $\wedge$ as defined in the usual way, i.e., $A \wedge B =_{\text{def.}} \neg(\neg A \vee \neg B)$. Using this definition, we can show:

Proposition 24 $\Gamma \mid\!\!\frac{\quad}{\mathsf{CL}^{+\backslash[cut]}} \Delta$ *iff* $\Gamma \mid\!\!\frac{\quad}{\mathsf{ST}} \Delta$.

Proof We can transform any proof-tree for $\Gamma \mid\!\!\frac{\quad}{\mathsf{CL}^{+\backslash[cut]}} \Delta$ into a proof-tree for $\Gamma \mid\!\!\frac{\quad}{\mathsf{ST}} \Delta$ and vice versa. Left-to-right: Applications of [ID] in $\mathsf{CL}^{+\backslash[cut]}$ can be translated into ST by [ID-ST] followed by [weakening-ST]. Applications of [weakening] are merely relabeled as applications of [weakening-ST]. Similarly for applications of the negation rules, truth rules, and [R$\vee$] (all in both directions). Top-to-bottom applications of [L$\vee$] are translated into applications of [L$\vee$-ST], leaving out the third top-sequent. Bottom-to-top applications of [L$\vee$] are translated into similar applications of [L$\vee$-ST] with an application of [weakening-ST] if the desired sequent is the third top-sequent. Top-to-bottom applications of [L$\wedge$] are translated by putting negations of both subaltern sentences on the right by [R$\neg$-ST], then using [R$\vee$-ST] to get their disjunction on the right, and finally using [L$\neg$-ST] to get the negation of the disjunction on the left. This negated disjunction is, by definition, the same as the desired conjunction. Bottom-to-top applications of [L$\wedge$] are translated by the same route in reverse. Top-to-bottom applications of [R$\wedge$] are translated by applying [L$\neg$-ST] to the first two top-sequents, omitting the third, then disjoining the resulting negations on the left via [L$\vee$-ST], and finally putting the negated disjunction on the right via [R$\neg$-ST]. This negated disjunction is, by definition, the same as the desired conjunction. Bottom-to-top applications of [R$\wedge$] are translated by the same route in reverse with an addition of [weakening-ST] if the desired sequent is the third top-sequent.

Right-to-left: Applications of [ID-ST], the ST rules for negation, truth, weakening, and [R$\vee$-ST] are translated by merely relabeling them appropriately. This leaves only [L$\vee$-ST]. Top-to-bottom applications are translated by [weakening] to get the required additional top-sequent followed by [L$\vee$]. Bottom-to-top applications can be merely relabeled. $\square$

Proposition 25 *If we add a truth-predicate to our language, the clauses (tr+) and (tr−) to our semantics, and a sentence* $\lambda = \neg Tr(\bar{\lambda})$, *then* $\models_{TM}$ *is trivial, i.e.,* $\forall\Gamma\forall\Delta\,(\Gamma \models_{TM} \Delta)$.

Proof The clauses for truth imply that a state that verifies the liar sentence also falsifies it, and vice versa. For suppose $s \Vdash \lambda$. Since $\lambda = \neg Tr(\bar{\lambda})$, it follows that $s \Vdash \neg Tr(\bar{\lambda})$. By (neg+), $s \dashv\!| \; Tr(\bar{\lambda})$. And by (tr−), $s \dashv\!| \; \lambda$. The same reasoning works in reverse. Now, let s be an arbitrary verifier of λ. Hence, $s \Vdash \lambda$ and $s \dashv\!| \; \lambda$. By Exclusivity, $\forall u(s \sqcup s \sqcup u \notin S^{\Diamond})$. So, $\forall u(s \sqcup u \notin S^{\Diamond})$. Now suppose for reductio that $\Gamma \not\models_{TM} \Delta$ and let u be a fusion of verifiers for each element in Γ and falsifiers for each element in Δ. By truth-maker bilateralism, $u \in S^{\Diamond}$. By Exhaustivity, there

[21] The proofs of these facts are straightforward, and I leave them as an exercise to the reader.

is either a verifier or a falsifier of λ that we can fuse with u into a possible state. Without loss of generality, let that state be s. Hence, $u \sqcup s \in S^{\Diamond}$, which contradicts the earlier result. So, by reductio, $\Gamma \models_{TM} \Delta$. □

In order to prove the completeness of $\mathsf{CL}^{+\backslash[cut]}$ with respect to $\models^{ST}_{TM}$, we cannot use the technique of proof-searches from above because proof-searches are no longer guaranteed to terminate. Hence, I follow Ripley [7, 162-63] in using the technique of (possibly infinite) reduction trees from Takeuti [41].[22]

Definition 14 (Reduction tree) The reduction tree for a sequent, $\Gamma \succ \Delta$, is the possibly infinite tree that results from starting with $\Gamma \succ \Delta$ as the root of the tree and then extending at each stage each top-most sequent of the tree as follows, until all branches are closed or else extending the tree ω-many times: (i) If the sequent is an axiom, i.e., is such that the left and the right side share an atomic sentence, then the branch remains unchanged and is closed. (ii) If the sequent has the form $\Gamma, \neg A \succ \Delta$ or $\Gamma \succ \neg A, \Delta$ and no reduction has been applied to $\neg A$ in previous stages, they reduce to $\Gamma, \neg A \succ A, \Delta$ and $\Gamma, A \succ \neg A, \Delta$ respectively. (iii) If the sequent has the form $\Gamma, A \wedge B \succ \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, it reduces to $\Gamma, A, B, A \wedge B \succ \Delta$; and if it has the form $\Gamma \succ A \wedge B, \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, the reduction tree branches into $\Gamma \succ A \wedge B, A, \Delta$ and $\Gamma \succ A \wedge B, B, \Delta$ and $\Gamma \succ A \wedge B, A, B, \Delta$. (iv) Similarly, $\Gamma \succ A \vee B, \Delta$ reduces to $\Gamma \succ A \vee B, A, B, \Delta$; and $\Gamma, A \vee B \succ \Delta$ reduces to $\Gamma, A, A \vee B \succ \Delta$ and $\Gamma, B, A \vee B \succ \Delta$ and $\Gamma, A, B, A \vee B \succ \Delta$. (v) $\Gamma, Tr(\bar{A}) \succ \Delta$ reduces to $\Gamma, A, Tr(\bar{A}) \succ \Delta$; and $\Gamma \succ Tr(\bar{A}), \Delta$ reduces to $\Gamma \succ Tr(\bar{A}), A, \Delta$.

Lemma 3 *The set of states deemed impossible by a sequent is the union of the states deemed impossible by the sequents to which it reduces in a reduction tree.*

Proof We look at each clause in the reduction procedure. The lemma holds trivially for clause (i). It holds for (ii) because the truth-makers of $\neg A$ are exactly the falsity-makers of A, and vice versa. The other cases, in particular those for (v), are analogous except for when the reduction tree branches out, such as in the case of $\Gamma \succ A \wedge B, \Delta$. In this case, the lemma holds because the falsity-makers of $A \wedge B$ are the union of the falsity-makers of A, the falsity-makers of B, and any fusion of such falsity-makers, which corresponds to the three sequents that result from the reduction. □

Definition 15 (Sequents resulting from an open branch of a reduction tree) If an open branch of a reduction tree terminates, the resulting sequent is the leaf of that branch. If the open branch does not terminate, then the resulting sequent is the sequent $\Gamma_\omega \succ \Delta_\omega$, where Γ_ω is the union of all the sets on the left side of sequents in this open branch and and Δ_ω is the union of the sets on the right side of sequents in the branch.

[22] Thanks to Lucas Rosenblatt for helpful feedback on this point.

Lemma 4 *Let $\Gamma \succ \Delta$ be a sequent resulting from an open branch, let Γ^{at} be the set of atomic sentences in Γ, and let Δ^{at} be the set of atomic sentences in Δ. Then a state that is deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ includes as a part a truth-maker for every sentence in Γ and a falsity-maker for every sentence in Δ.*

Proof We argue by induction on the complexity of sentences in $\Gamma \cup \Delta$. The states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ trivially include truth-makers for every atomic sentence in Γ and falsity-makers for every atomic sentence in Δ. Suppose our lemma holds for sentence up to complexity n, and let's consider sentences of complexity $n + 1$. Note that since we have an open branch, we know that all possible reduction procedures have been applied. For negations in Γ, we know that the negatum, which is of complexity n, is in Δ. So, by our induction hypothesis states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ include a falsity-maker for the negatum, which is a truth-maker for our negation. Similarly for all other connectives where the reduction tree does not branch. For disjunctions on the left, we know that Γ contains also one or both of the disjunctions, which are of complexity n. So by our hypothesis, $\Gamma^{at} \succ \Delta^{at}$ contains truth-makers for one or both disjuncts, and any of these options ensures that it includes a truth-maker for the disjunction. Similarly for conjunctions on the right. □

Proposition 26 $\Gamma \mathrel{|}\!\underline{}_{\mathsf{CL}^{+\backslash[cut]}} \Delta$ *iff* $\Gamma \models^{ST}_{TM} \Delta$.

Proof Left-to-right: We leave out [cut] and Exhaustivity in Propositions 20, and the proof still shows the validity of [ID] and [weakening]. Since the operational rules haven't changed, the validity proof for the operational rules from Proposition 21 still applies. Clauses (tr+) and (tr−) ensure that [Lt] and [Rt] are valid for $\models^{ST}_{TM}$.

Right-to-left: Suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a reduction tree for $\Gamma \succ \Delta$ has an open branch. Let $\Gamma_\omega \succ \Delta_\omega$ be the sequent that results from that branch, and let $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ be the sequent that results from $\Gamma_\omega \succ \Delta_\omega$ by omitting all complex sentences. We can use as our desired counter-model any model that makes possible one of the states deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$. For by Lemma 4, any state that is deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ is also deemed impossible by $\Gamma_\omega \succ \Delta_\omega$. And by Lemma 3, any state deemed impossible by $\Gamma_\omega \succ \Delta_\omega$ is deemed impossible by $\Gamma \succ \Delta$. So our model is a counter-model to $\Gamma \succ \Delta$. We know that there is such a model because any model will work that makes only those states impossible that are required to be impossible by Exclusivity, and $\Gamma^{at}_\omega \cap \Delta^{at}_\omega = \emptyset$. □

Note that the models that make possible all or some of the states deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ can respect our semantic clauses for the transparent truth-predicate, even though the atomic sentences in $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ contain truth-predicates applied to complex sentences. For if, e.g., $Tr(\overline{A \wedge B}) \in \Gamma^{at}_\omega$, then $A \wedge B \in \Gamma_\omega$. So the atomic part of the reduction of $A \wedge B$ is also in Γ^{at}_ω. And if A or B include truth-predications, this will apply to those again.

For the relation to LP, we will use the following known fact [24, 25]:

Fact 2 $\Gamma \models_{LP} \Delta$ iff $\succ\Delta$ is provable if we add $\succ\gamma$ for all $\gamma \in \Gamma$ as axioms to the sequent calculus for ST.

Proposition 27 *$\Gamma \models_{LP} \Delta$ iff $\succ\Delta$ is provable if we add $\succ\gamma$ for all $\gamma \in \Gamma$ as axioms to* $\mathsf{CL}^{\backslash[cut]}$.

Proof We have seen in the proof for Proposition 24 that we can transform every ST proof-tree into a $\mathsf{CL}^{+\backslash[cut]}$ proof-tree and vice versa. □

Proposition 28 *$\Gamma \models_{LP} \Delta$ iff no fusion of falsifiers of everything in Δ is possible if no falsifier of anything in Γ is possible, i.e., $\forall d \in |\bigvee \Delta|^- (d \notin S^\Diamond)$ in all $\models^{ST}_{TM}$ models in which $\forall\gamma \in \Gamma\ (\forall g \in |\gamma|^- (g \notin S^\Diamond))$.*

Proof Left-to-right: Suppose that $\Gamma \models_{LP} \Delta$. By Proposition 27, $\succ\Delta$ is provable if we add $\succ\gamma$ for all $\gamma \in \Gamma$ as axioms to $\mathsf{CL}^{+\backslash[cut]}$. To add $\succ\gamma$ for all $\gamma \in \Gamma$ as axioms to $\mathsf{CL}^{+\backslash[cut]}$ corresponds in our semantics to the elimination of all models in which any falsifier of any member of Γ is a possible state. Thus, we only look at models in which $\forall\gamma \in \Gamma\ (\forall g \in |\gamma|^- (g \notin S^\Diamond))$. Since $\succ\Delta$ is provable and, by Proposition 26, our sequent rules are sound for $\models^{ST}_{TM}$, it follows that in all such models the fusion of any falsifiers of the members of Δ is impossible, i.e., $\forall d \in |\bigvee \Delta|^- (d \notin S^\Diamond)$.

Right-to-left: Suppose that $\forall d \in |\bigvee \Delta|^- (d \notin S^\Diamond)$ in all models in which $\forall\gamma \in \Gamma\ (\forall g \in |\gamma|^- (g \notin S^\Diamond))$. By the completeness of $\mathsf{CL}^{+\backslash[cut]}$ for $\models^{ST}_{TM}$, this means that $\succ\Delta$ is provable if we add $\succ\gamma$ for all $\gamma \in \Gamma$ as axioms to $\mathsf{CL}^{+\backslash[cut]}$. So, by Proposition 27, $\Gamma \models_{LP} \Delta$. □

Proposition 29 *$\Gamma \models_{K3} \Delta$ iff $\Gamma\succ$ is provable if we add $\delta\succ$ for all $\delta \in \Delta$ as axioms to* $\mathsf{CL}^{+\backslash[cut]}$.

Proof Suppose that $\Gamma \models_{K3} \Delta$. By the duality of $\mathsf{K3}$ and LP, it follows that $\neg\Delta \models_{LP} \neg\Gamma$, where $\neg X$ is the set of the negations of the members of X. By Proposition 27, it follows that $\succ\neg\Gamma$ is provable if we add $\succ\neg\delta$ for all $\delta \in \Delta$ as an axiom to $\mathsf{CL}^{+\backslash[cut]}$. Given the negation rules of $\mathsf{CL}^{+\backslash[cut]}$ this means that $\Gamma\succ$ is provable if we add $\delta\succ$ for all $\delta \in \Delta$ as an axiom to $\mathsf{CL}^{+\backslash[cut]}$. For the other direction, the same reasoning works in reverse. □

Proposition 30 *$\Gamma \models_{K3} \Delta$ iff $\forall g \in |\bigwedge \Gamma|^+ (g \notin S^\Diamond)$ in all $\models^{ST}_{TM}$ models in which $\forall\delta \in \Delta\ (\forall d \in |\delta|^+ (d \notin S^\Diamond))$.*

Proof The proof is analogous to the one for LP above. □

Proposition 31 *$\Gamma \vdash_{\mathsf{CL}^{\backslash[ID]}} \Delta$ iff $\Gamma \models_{TS} \Delta$ iff $\Gamma \models^{TS}_{TM} \Delta$.*

Proof It is well known that $\models_{TS}$ is empty. No sequent is derivable in $\mathsf{CL}^{\backslash[ID]}$ because there is no way to start proofs without [ID]. And $\models^{TS}_{TM}$ is empty because we can let all states be possible. □

Proposition 32 *Let $\mathfrak{I}$ be a set of pairs, $\langle \Gamma, \Delta \rangle$, where Γ and Δ are sets of sentences. Regardless of whether we close $\mathfrak{I}$ under the rules of* $\mathsf{CL}^{\backslash[ID]}$ *, of* TS*, or of* $\models^{TS}_{TM}$*, the resulting consequence relation is the same.*

Proof Since the correspondences are familiar by now, I give merely some quick pointers. As we have seen in the proofs of Lemma 2 and 24 the rules governing the logical connectives in all three systems are equivalent, given [weakening]. Exhaustivity in $\models^{TS}_{TM}$ corresponds to [cut] in $\mathsf{CL}^{\backslash[ID]}$ and TS, and Downward-Closure corresponds to [weakening] (see Proposition 20 above). □

A.3 Relation of CL to Correia's Logic, Containment, and Entailment

Proposition 33 *The leaves of a proof-tree that result from a proof-search on $A \succ$ are such that the union of the states that they deem impossible is exactly the set of truth-makers of A.*

Proof A tree that results from a proof-search uses only top-to-bottom applications of operational rules. By Lemma 2, the union of the states deemed impossible by the top-sequents of such a rule-application is the set of states deemed impossible by the bottom sequent. Hence, for any proof-tree that results from a proof-search, the union of the states deemed impossible by the leaves of the tree is the set of states deemed impossible by the root. The set of states deemed impossible by $A \succ$ are exactly the truth-makers of A. □

As Correia [29] shows, the following fact holds:

Fact 3 $A \approx B$ is a theorem of Correia's logic iff A and B have the same truth-makers in every truth-maker model.

Proposition 34 *$A \approx B$ iff proof-searches on $A \succ$ and $B \succ$ yield the same result.*

Proof $A \approx B$ holds iff A and B have the same truth-makers in all models. By Proposition 33, this holds just in case proof-searches in CL on $A \succ$ and $B \succ$ yield the same result. □

Since proof-searches use only the operational rules of CL, this last result immediately implies our target:

Proposition 35 *$A \approx B$ is a theorem of Correia's logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B \succ \Delta$ and vice versa are admissible.*

Fact 4 As Correia [29, 117] shows, $A \approx B$ is provable in his dual system iff $\neg A \approx \neg B$ is provable in his original system.

Proposition 36 *$A \approx B$ is a theorem of Correia's dual logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible iff A and B have the same falsity-makers in all models.*

Proof Suppose that $A \approx B$ is a theorem of Correia's dual logic. By Fact 4 and Proposition 35 and the negation rules of CL, this happens iff the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible. And this happens just in case proof-searches for $\succ A$ and $\succ B$ yield the same atomic sequents. By reasoning that is parallel to the proof of Proposition 33, this happens iff A and B have the same falsity-makers in all models. □

Proposition 37 *A contains B in virtue of logical form iff, for some Θ, the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible.*

Proof Left-to-right: Suppose that A contains B. By definition, there is a proposition R such that $|A|^+ = \{b \sqcup r : b \in |B|^+ \text{ and } r \in |R|^+\}$. In accordance with Assumption 1, let Θ be a set of sentences such that the set of fusions of verifiers for each of the elements is $|R|^+$. Hence, A and $B \wedge \bigwedge \Theta$ have the same verifiers. Therefore, by Proposition 33, a proof-search on $A \succ$ and on $B \wedge \bigwedge \Theta \succ$ yield the same result. This ensures that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible.

Right-to-left: Suppose the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible. This happens only if proof-searches on $A \succ$ and on $B, \Theta \succ$ yield the same result. By Proposition 33, it follows that $\{t \sqcup p : t \in |\bigwedge \Theta|^+ \text{ and } b \in |B|^+\} = |A|^+$. So every verifier of A includes a verifier of B as a part, and every verifier of B is included as a part in a verifier of A. Therefore, A contains B in virtue of logical form. □

Proposition 38 *A entails B in virtue of logical form iff the operational rules of* CL *suffice to show that the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible.*

Proof Left-to-right: Suppose that A entails B in virtue of logical form and, hence, in all models. Then the verifiers of A are a subset of the verifiers of B. So, by Proposition 33, the union of the states deemed impossible by the leaves of the proof-tree for $A \succ$ is a subset of the union of the states deemed impossible by the leaves of the proof-tree for $B \succ$. And this holds in all models. Suppose for reductio that there is a leaf, $\Gamma_0 \succ \Delta_0$, in the proof-tree for $A \succ$ that is not also a leaf in the tree for $B \succ$. We take a model in which s is the unique state that is deemed impossible by $\Gamma_0 \succ \Delta_0$, and we ensure that s is not deemed impossible by any of the leaves in the tree for $B \succ$. Then s is a verifier of A but not of B, contradicting our assumption that A entails B. But if the leaves of the proof-tree for $A \succ$ is a subset of those for $B \succ$, then the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible.

Right-to-left: Suppose that the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible. Then proof-searches on $A \succ$ and on $B \succ$ yield proof-trees such that the leaves of the tree for $A \succ$ is a subset of the leaves of the tree for $B \succ$. This must hold in virtue of logical form. The union of the states deemed impossible by the leaves of the tree for

$A\succ$ is a subset of the corresponding states for $B\succ$. By Proposition 33, every verifier of A is a verifier of B. So, A entails B in virtue of logical form. □

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